AM-GM inequality

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Question 1

(a) Let $f(x) = \ln x - x + 1, \quad x > 0$. Prove that $f(x) \le 0, \quad \forall\ x > 0$.

(b) Let $a_1, a_2, \ldots, a_n > 0$. $A = \dfrac{a_1 + a_2 + \ldots + a_n}{n}, G = \sqrt[n]{a_1 a_2 \ldots a_n}$

Putting $x = \dfrac{a_i}{A}$ in **(a)**, prove that

$A \ge G$. The equality sign holds if $a_1 = a_2 = \ldots = a_n$.

(c) Use **(b)** to prove that $\left[\sum_{i=1}^{n} a_i\right]\left[\sum_{i=1}^{n} \dfrac{1}{a_i}\right] \ge n^2$ and hence show that

$$1 + \frac{1}{2} + \ldots + \frac{1}{n} \ge \frac{2n}{n+1}$$

Solution

(a) $f(x) = \ln x - x + 1, \quad x > 0$

$$f'(x) = \frac{1}{x} - 1 = 0 \quad \text{when} \quad x = 1.$$

When $0 < x < 1$, $f'(x) > 0$.

When $x > 1$, $f'(x) < 0$

$\therefore$ f(x) attains its maximum value at $x = 1$.

$\therefore$ $f(x) \le f(1) = \ln 1 - 1 + 1 = 0 \quad \forall\ x > 0$.

$\therefore$ $f(x) \le 0 \quad \forall\ x > 0$.

(b) Let $A = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$.

Then for $i = 1, 2, \ldots, n$, $f\left(\dfrac{a_i}{A}\right) = \ln\left(\dfrac{a_i}{A}\right) - \dfrac{a_i}{A} + 1 \le 0$, $\ln\left(\dfrac{a_i}{A}\right) \le \dfrac{a_i}{A} - 1$

$$\therefore \quad \sum_{i=1}^{n} \ln\left(\frac{a_i}{A}\right) \le \sum_{i=1}^{n}\left(\frac{a_i}{A} - 1\right)$$

$$\ln\left[\frac{a_1 a_2 \ldots a_n}{A^n}\right] \le \frac{1}{A}\sum_{i=1}^{n} a_i - n = \frac{1}{A}(nA) - n = 0$$

$$\frac{a_1 a_2 \ldots a_n}{A^n} \le 1$$

$$\therefore G = \sqrt[n]{a_1 a_2 \ldots a_n} \le A$$

The equality holds if $\dfrac{a_i}{A} = 1, \quad \forall i, \quad 1 \le i \le n$.

i.e. $a_1 = a_2 = \ldots = a_n = A$.

(c) $\dfrac{a_1+a_2+....+a_n}{n} \ge \sqrt[n]{a_1a_2....a_n}$

$\therefore \quad \sum_{i=1}^{n} a_i \ge n\sqrt[n]{a_1a_2....a_n}$ ….(1)

Replace a_i by $\dfrac{1}{a_i}$ in (1), we have

$\sum_{i=1}^{n} \dfrac{1}{a_i} \ge n\sqrt[n]{\dfrac{1}{a_1a_2....a_n}}$ ….(2)

(1) × (2), we get $\left[\sum_{i=1}^{n} a_i\right]\left[\sum_{i=1}^{n}\dfrac{1}{a_i}\right] \ge n^2$ ….(3)

Put $a_i = i$ in (3), $\left[\sum_{i=1}^{n} i\right]\left[\sum_{i=1}^{n}\dfrac{1}{i}\right] \ge n^2$

$$\frac{n(n+1)}{2}\left[\sum_{i=1}^{n}\frac{1}{i}\right] \ge n^2 \Rightarrow 1+\frac{1}{2}+....+\frac{1}{n} \ge \frac{2n}{n+1}$$

Question 2

By applying A.M. ≥ G.M., find the greatest value of each of the following functions and the value of x in which the maximum is attained:

(a) $f(x) = (1+2x)^3(1-3x)^2$, where $-\dfrac{1}{2} \le x \le \dfrac{1}{3}$

(b) $g(x) = (1+2x)^4(1-3x)^2$, where $-\dfrac{1}{2} \le x \le \dfrac{1}{3}$.

Solution

(a) $-\dfrac{1}{2} \le x \le \dfrac{1}{3} \Rightarrow 1 + 2x \ge 0$ and $1 - 3x \ge 0$.

Apply A.M. ≥ G.M., $\sqrt[5]{(1+2x)^3(1-3x)^2} \le \dfrac{3(1+2x)+2(1-3x)}{5} = 1 \Rightarrow f(x) \le 1$.

$\therefore$ Greatest value of $f(x)$ is 1 which is attained when:

$1 + 2x = 1 - 3x \Rightarrow x = 0$.

(b) Apply A.M. ≥ G.M. to six numbers : four $\dfrac{3}{4}(1+2x)$ and two $(1-3x)$

$$\left[\frac{3}{4}(1+2x)\right]^4(1-3x)^2 \le \left[\frac{4\times\frac{3}{4}(1+2x)+2\times(1-3x)}{6}\right]^6$$

$$\therefore \quad \frac{3^4}{4^4}g(x) \le \frac{5^6}{6^6} \Rightarrow g(x) \le \frac{5^6}{6^6}\times\frac{4^4}{3^4} = \frac{5^6\times 2^2}{3^{10}} = \frac{62500}{59049}$$

which is attained when

$\dfrac{3}{4}(1+2x) = 1 - 3x \Rightarrow x = \dfrac{1}{18}$.

Question 3

(a) By considering $\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$, factorize $a^3 + b^3 + c^3 - 3abc$ in two real factors.

(b) Use **(a)** to show that if a, b, c are positive, $a^3 + b^3 + c^3 \geq 3abc$.

(c) Use **(b)** to prove that if x, y, z are positive, $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

(d) Prove that $(p+q+r)\left(\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}\right) \geq 9$, where p, q, r are positive.

(e) Prove that if α, β, γ are three sides of a triangle,

$$\frac{1}{\alpha+\beta-\gamma}+\frac{1}{\beta+\gamma-\alpha}+\frac{1}{\gamma+\alpha-\beta} \geq \frac{9}{\alpha+\beta+\gamma}$$

Solution

(a) $\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$,on direct expansion using Sarrus rule.

$$\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \overset{C_1 \to C_1+C_2+C_3}{=} \begin{vmatrix} a+b+c & c & b \\ a+b+c & a & c \\ a+b+c & b & a \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & c & b \\ 1 & a & c \\ 1 & b & a \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$\therefore \quad a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

(b) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= (a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \geq 0$, since a, b, c ≥ 0.

$\therefore \quad a^3 + b^3 + c^3 \geq 3abc$.

(c) Put $x = a^3$, $y = b^3$, $z = c^3$ in **(b)**, $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$

(d) $(p+q+r)\left(\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}\right) = 9\left(\dfrac{p+q+r}{3}\right)\left(\dfrac{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}}{3}\right) \geq 9\sqrt[3]{pqr}\sqrt[3]{\dfrac{1}{pqr}} = 9$, by **(c)**

(e) Since α, β, γ are three sides of a triangle, $\alpha + \beta - \gamma$, $\beta + \gamma - \alpha$, $\gamma + \alpha - \beta \geq 0$.

In **(d),** put $p = \alpha + \beta - \gamma$, $q = \beta + \gamma - \alpha$, $r = \gamma + \alpha - \beta$ in **(d)** ,

$p + q + r = \alpha + \beta + \gamma$, $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r} = \dfrac{1}{\alpha+\beta-\gamma}+\dfrac{1}{\beta+\gamma-\alpha}+\dfrac{1}{\gamma+\alpha-\beta}$

$$\therefore \quad \frac{1}{\alpha+\beta-\gamma}+\frac{1}{\beta+\gamma-\alpha}+\frac{1}{\gamma+\alpha-\beta} \geq \frac{9}{\alpha+\beta+\gamma} \quad .$$